Quantum state diffusion, localization and quantum dispersion entropy

1993 J. Phys. A: Math. Gen. 26 2233

(http://iopscience.iop.org/0305-4470/26/9/018)

J. Phys. A: Math. Gen. 26 (1993) 2233–2243. Printed in the UK

Quantum state diffusion, localization and quantum dispersion entropy

Nicolas Gisin† and Ian C Percival‡

† Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland
‡ Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, UK and Department of Physics, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, UK

Received 8 December 1992

Abstract. The quantum state diffusion model introduced in an earlier paper represents the evolution of an individual open quantum system by an Itô diffusion equation for its quantum state. The diffusion and drift terms in this equation are derived from interaction with the environment. In this paper two localization theorems are proved. The dispersion entropy theorem shows that under special conditions, which are commonly satisfied to a good approximation, the mean quantum dispersion entropy, which measures the mean dispersion or delocalization of the quantum states, decreases at a rate equal to a weighted sum of effective interaction rates, so that the localization always increases in the mean, except when the effective interaction with the environment is zero. The general localization theorem provides a formula for more general conditions.

1. Introduction

The motion of an ensemble of Brownian particles starting at the same initial point can be represented by a Gaussian probability distribution with mean squared deviation that increases linearly with time. This averaged picture contrasts with the more detailed explicit picture of a single particle that moves along a very crooked path in space, according to a simple stochastic Itô equation.

In [1] we introduced a diffusion model for the quantum state of an open quantum system, which is summarized in section 2, and showed how this model could be applied to investigate various physical processes. As for the Brownian particle, the solution of the Itô equation for the state vector for an individual system gives a more detailed explicit picture of these processes than the usual picture based on the evolution of the density operator for an ensemble of systems. This diffusion model is an extension of earlier work by Gisin [2, 3] and Diósi [4].

What the density operator gains in mathematical elegance it loses in physical directness. Instead of looking at the deterministic evolution of the density operator ρ representing an ensemble of systems, we look at the stochastic diffusion of a quantum state representing an individual system of the ensemble, as shown explicitly in the illustrations of [1]. This picture of the evolution of an individual system is particularly advantageous in the theories of absorption and measurement. *Localization* is a stochastic process caused by the interaction of the system with its environment, which tends to concentrate the diffusing state vector in a particular subspace or *channel* of the state space, whether or not there is a measuring device. Localization in position or configuration space is one important example, but it is

0305-4470/93/092233+11$07.50

not the only one. We use *localization*, rather than *reduction* because we need to refer to the condition of being localized, and not just the process by which it takes place.

In this paper we prove two basic theorems on localization for the state diffusion model and, in the following paper, we show how to apply the model to a wide range of physical processes for which localization is important.

In section 3 we define the quantum dispersion entropy, which is a measure of the dispersion or delocalization of a pure quantum state. We then state and briefly discuss the two localization theorems, both of which are proved in section 4. The *dispersion entropy theorem* applies under restricted conditions and states that the mean rate of decrease in the dispersion entropy of quantum states is equal to a sum over weighted effective interaction rates. Consequently the mean over the ensemble of the localization never decreases under these conditions. The *general localization theorem* provides a formula for less restrictive conditions.

The final section provides a discussion and comparison with other theories, including those of Zeh *et al* [5], Zurek [6], Ghirardi *et al* [7], Dalibard, Carmichael [8] and their collaborators. Detailed studies of particular physical processes are left to the next paper.

The space of states $|\psi\rangle$ of a quantum system may be divided or *partitioned* into orthogonal subspaces or *channels*. Each such division into channels is a partition of the state space. The channels are labelled by the projectors P_k that project onto them, where

$$P_k P_\ell = P_\ell P_k = \delta_{k\ell} P_\ell \qquad \sum_k P_k = I. \tag{1.1}$$

These parts are often *spatially* separated and this is the most important example. It applies, for example, to experiments with spatially separated channels, like the Stern–Gerlach experiment.

A pure quantum state has quantum statistical properties with respect to these channels. In particular, each projection operator has a mean or expectation and a mean square deviation, and the state has an entropy which measures the dispersion or delocalization among the channels. These are the *quantum* expectations, quantum mean square (QMS) deviations and the quantum dispersion entropy. For ensembles of quantum states there are also *ensemble* means, represented by M, ensemble mean square deviations and ensemble entropies, with properties very different from the quantum versions. There are also combinations of quantum and ensemble statistical properties. These distinctions are necessary for state diffusion theory. Because of the consistency of the state diffusion theory and the orthodox theory using density operators, ensemble means of the quantum probabilities are the same as the means obtained from density operators in the usual way. However, many quantities that can be defined when quantum and ensemble statistical properties are separated cannot be defined using the density operators alone, for example the ensemble mean of the quantum mean square variation or the ensemble mean of the quantum dispersion entropy.

As for Brownian motion, the dynamics of arbitrary initial distributions can be expressed in terms of the dynamics of an initial δ-distribution in which all members of the ensemble have the same initial quantum state $|\psi(0)\rangle$. All state vectors are supposed normalized. We state the two theorems for this special case. The generalization to an arbitary initial distribution is immediate, by taking a mean over the initial distribution.

The quantum expectation of the projector P:

$$\langle\psi|P|\psi\rangle = \langle P\rangle_\psi = \langle P\rangle = p \tag{1.2}$$

is the probability of the system in state $|\psi\rangle$ being in the channel P. Unless otherwise specified all expectations refer to the state $|\psi\rangle$. The extent to which a state $|\psi\rangle$ is delocalized in the subspace of P or of its complementary projector $I - P$ can be measured by the QMS deviation

$$(\Delta p)^2 = \langle P^2 \rangle - \langle P \rangle^2 = \langle P \rangle - \langle P \rangle^2 = \langle P \rangle \langle I - P \rangle = p(1-p) \tag{1.3}$$

which is symmetric in P and its complement.

The extent to which the state $|\psi\rangle$ is delocalized or dispersed for an arbitrary partition into channels P_k is conveniently measured by the quantum *dispersion entropy* of the state with respect to the partition, given by

$$Q = Q(|\psi\rangle, P_k) = -\sum_k p_k \ln p_k \tag{1.4}$$

where the quantum probability p_k is given by

$$p_k = \langle P_k \rangle_\psi = \langle \psi | P_k | \psi \rangle. \tag{1.5}$$

The system localizes with respect to the channels P and $I - P$ or the channels P_k when the QMS deviation or the quantum dispersion entropy *decreases* with time.

Depending on the nature of the system and its interaction with the environment, localization may take place with respect to many different variables, but localization in position is particularly important because interactions are localized in position space to a greater extent than for other dynamical variables. Relativistically, the same localization is a consequence of the impossibility of direct interaction over spacelike intervals [3, 9].

The 'environment' in our theory is not necessarily separated from the system in position space. For example, an electromagnetic field in a cavity can act as an environment for the atoms in the cavity and *vice versa*. An example of the first is the theory of radiative transitions of atoms, and the theory of gas lasers is an example of the second.

In quantum state diffusion theory it is helpful to consider the following principles:

Principle 1. Interactions with independent parts of the environment lead to statistically independent quantum state diffusions.

Principle 2. *Amplitudes* fluctuate where *probabilities* are conserved: where there is an interaction with the environment in which the overall probability of being in a subspace of the state space is conserved, the corresponding amplitudes of the state of individual systems of the ensemble fluctuate, unless the effective rate of interaction of the system with its environment is zero.

Principle 3. Probability conservation and fluctuation together imply localization (with special exceptions).

Principle 4. Localization of states in position space follows from locality of interactions. The rest frame of the localization is determined by the environment.

These principles are discussed in more detail in section 3.

2. Quantum state diffusion

Quantum state diffusion represents the evolution of a quantum system in interaction with its environment through a unique correspondence between the deterministic Bloch equation for the ensemble density operator ρ and an Itô diffusion equation for the normalized state vector $|\psi\rangle$ of an individual system of the ensemble [1, 2, 4]. The correspondence ensures that the diffusion equation and the Bloch equation give the same physics. If H is the Hamiltonian representing the dynamics of an open system, and L_m are the environment operators which represent the interaction of the system with its environment, then the Bloch equation is

$$\dot{\rho} = -\frac{\mathrm{i}}{\hbar}[H, \rho] + \sum_m (L_m \rho L_m^\dagger - \tfrac{1}{2} L_m^\dagger L_m \rho - \tfrac{1}{2} \rho L_m^\dagger L_m). \tag{2.1}$$

The corresponding quantum state diffusion equation is a stochastic differential equation for the normalized state vector $|\psi\rangle$, whose differerential Itô form is

$$|\mathrm{d}\psi\rangle = -\frac{\mathrm{i}}{\hbar} H|\psi\rangle\,\mathrm{d}t + \sum_m (\langle L_m^\dagger\rangle_\psi L_m - \tfrac{1}{2} L_m^\dagger L_m - \tfrac{1}{2}\langle L_m^\dagger\rangle_\psi \langle L_m\rangle_\psi)|\psi\rangle\,\mathrm{d}t + \sum_m (L_m - \langle L_m\rangle_\psi)|\psi\rangle\,\mathrm{d}\xi_m \tag{2.2}$$

where $\langle L_m\rangle_\psi = \langle\psi|L_m|\psi\rangle$ is the expectation of L_m for state $|\psi\rangle$ and the density operator is given by the mean over the projectors onto the quantum states of the ensemble,

$$\rho = \mathrm{M}|\psi\rangle\langle\psi| \tag{2.3}$$

where M represents a mean over the ensemble.

The first sum in (2.2) represents the 'drift' of the state vector and the second sum the random fluctuations, both due to the interaction of the system with its environment. The $\mathrm{d}\xi_m$ are independent complex differential random variables, each of which has normalized independent white noise in its real and imaginary parts, leading to an isotropic Brownian motion or Wiener process in the complex ξ_m-plane, satisifying the complex relations

$$\mathrm{M}\,\mathrm{d}\xi_m = 0 \tag{2.4a}$$

$$\mathrm{M}(\mathrm{d}\xi_n\,\mathrm{d}\xi_m) = 0 \qquad \mathrm{M}(\mathrm{d}\xi_n^*\,\mathrm{d}\xi_m) = \delta_{nm}\,\mathrm{d}t. \tag{2.4b}$$

The complex Wiener process is normalized to unity, so the independent real and imaginary Wiener processes are each normalized to $\frac{1}{2}$. Notice that this is a different normalization from that of [1], and that there is also a different normalization for the operators L in equations (2.1) and (2.2).

The environment operators L_m of the Bloch equation are obtained by taking a trace over the states of the environment, assuming that the time constants for the interaction with the environment are sufficiently short for the evolution of the system to be represented by a Markov process. Like the Bloch equation, the state diffusion equation is dependent on this approximation. They are also both dependent on the choice of the boundary between the system and its environment, but in many circumstances a choice can be made so that the dependence is weak.

There is a close analogy between the state diffusion theory of quantum mechanics and classical circuit theory. An electrical circuit includes both Hamiltonian elements, with capacitance and inductance, and non-Hamiltonian elements, such as resistors, which represent the interaction of the system with its environment. Similarly the environment operators of the state diffusion theory represent particular types of interaction of the system with its environment. There is no need to carry out a detailed physical analysis of these interactions before attempting to solve a problem in the theory of open quantum systems, as shown by example in [1].

3. Localization theorems

The theorems are the *dispersion entropy theorem* and the *general localization theorem*. Suppose there are two or more channels defined by projectors P_k satisfying (1.1), and we use a representation in which every base vector lies entirely in one channel. A *general* operator has non-zero off-diagonal elements which couple the channels. A *block diagonal operator* has no such non-zero elements, so that the channels are uncoupled. Even more special is a *local* operator which consists of only one diagonal block, with non-zero elements in one channel only. In the dispersion entropy theorem the Hamiltonian is block diagonal and the environment operators are local. In the less powerful general localization theorem the Hamiltonian is general or block diagonal and the environment operators are block diagonal.

For the dispersion entropy theorem the Hamiltonian satisfies

$$[H, P_k] = 0 \tag{3.1}$$

and every environment operator is confined to one of the channels. It is then denoted L_{kj}, where

$$L_{kj} = P_k L_{kj} P_k. \tag{3.2}$$

This is the case of *separable channels*, since each channel and its environment then operate independently. Neither the Hamiltonian H nor the environment operators L_{kj} couple the channels, so the ensemble mean $\mathrm{M}\langle P_k\rangle$ is conserved.

A normalized state vector $|\psi\rangle$ for the whole system can then be represented as a linear combination of normalized state vectors $|\psi_k\rangle$ for each channel with coefficients a_k, so that

$$|\psi\rangle = \sum_k a_k |\psi_k\rangle \tag{3.3}$$

where

$$a_k|\psi_k\rangle = P_k|\psi\rangle \tag{3.4}$$

$$|a_k|^2 = \langle P_k\rangle_\psi = p_k. \tag{3.5}$$

The expectations of the interaction operators are then

$$\langle L_{kj}\rangle = \langle L_{kj}\rangle_\psi = \langle\psi|L_{kj}|\psi\rangle \tag{3.6}$$

The *effective interaction rate* in channel k, is then defined as

$$R_k = \sum_j |\langle L_{kj}\rangle_\psi|^2 \tag{3.7}$$

which has the dimensions of inverse time.

It is convenient to define the magnitude of the dispersion or delocalization for state $|\psi\rangle$ in terms of the quantum dispersion entropy Q for the partition

$$Q = Q(|\psi\rangle, P_k) = -\sum_k p_k \ln p_k. \tag{3.8}$$

Evidently the smaller the entropy the greater the localization is, so an increase in mean localization is represented by a decrease in mean dispersion entropy.

The *dispersion entropy theorem* then states that the mean rate of decrease of quantum dispersion entropy for the partition is equal to a weighted sum over effective interaction rates.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{M}Q) = -\sum_k \frac{1-p_k}{p_k} R_k = -\sum_k p_k(1-p_k)(R_k/p_k^2) \leqslant 0 \tag{3.9}$$

where the second equality is expressed in terms of non-singular normalized interaction rates R_k/p_k^2. Consequently under the conditions of the theorem, the mean of the dispersion entropy never increases.

In the important case of only two complementary channels,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{M}Q) = -\left(\frac{p_2 R_1}{p_1} + \frac{p_1 R_2}{p_2}\right) = -p_1 p_2(R_1/p_1^2 + R_2/p_2^2) \leqslant 0. \tag{3.10}$$

The mean quantum dispersion entropy always decreases, so the localization always increases, unless the effective interaction rate is zero or the localization is complete. Some care is needed in applying this principle. For a non-zero interaction it is necessary that the system affects the environment. So the scattering of a particle by a fixed scattering centre has zero effective interaction rate, since there is zero recoil, and if the scatterer is very heavy, the effective interaction rate is very small. These cases can be represented exactly or approximately by an additional Hamiltonian term, not by environment operators. The same goes for the focusing and diffraction of photons by non-absorptive optical apparatus, or the electrons in an electron microscope, where the recoil of the apparatus is negligible.

If the effective interaction stops, as, for example, when dissipation produces a permanent ground state, then so does the localization. For a number of similar interactions in a single channel, the effective interaction rate is given by the sum of the rates for each interaction.

In the general localization theorem, the Hamiltonian can couple the channels and the environment operators L_m are block diagonal in the subspaces, so that

$$P_k L_m = L_m P_k = P_k L_m P_k. \tag{3.11}$$

The *general localization theorem* for an intital pure state then says that for each projector P_k the corresponding ensemble mean of the QMS deviation is

$$\mathrm{dM}(\Delta p_k)^2 = (1 - 2\langle P_k\rangle)\left\langle\frac{\mathrm{i}}{\hbar}[H, P_k]\right\rangle \mathrm{d}t - 2\sum_m |\langle P_k L_m\rangle - \langle P_k\rangle\langle L_m\rangle|^2 \,\mathrm{d}t. \tag{3.12}$$

If the Hamiltonian, like the environment operators, is block diagonal in the subspaces of P_k, then the first term is zero, and the system weakly localizes in the space of every P_k:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{M}(\Delta p_k)^2 \leqslant 0 \tag{3.13}$$

with equality when

$$\langle P_k L_m \rangle = \langle P_k \rangle \langle L_m \rangle \qquad \text{for all } m \tag{3.14}$$

which is a very special case.

The general localization theorem for an initial ensemble of states is obtained by taking the ensemble mean of (3.12) on both sides.

Now consider the four principles given in the introduction. There is no need for a measurement to take place for there to be localization. Independent parts of the environment give different sets of environment operators L_m, and by the quantum state diffusion equation (2.2) these lead to diffusions proportional to different $d\xi_m$, which are statistically independent. This is the principle 1, which can be applied to the diffusion of a particle by a gas or the absorption of a particle by a screen.

Principle 2 follows directly from the state diffusion equation (2.2) for a given $|\psi\rangle$. Amplitude fluctuations always occur unless

$$L_m|\psi\rangle = \langle L_m \rangle |\psi\rangle \qquad \text{for all } m \tag{3.15}$$

which is clearly exceptional. If the Hamiltonian H and the environment operators L_m are block diagonal, so that they do not couple the subspace of the projector P and its complement, the ensemble probability of being in either subspace is conserved, but the magnitude and phase of the amplitudes $P|\psi\rangle$ and $(I - P)|\psi\rangle$ both fluctuate, unless (3.15) is satisfied.

Principle 3 now follows directly from the theorems.

Now consider the special case of position localization, as in principle 4. In this case the projectors P_k project onto mutually exclusive regions of position space, and provided these regions are not too small, the coupling between the different parts of the environment in the different regions can be neglected. Furthermore, it is a consequence of the approximate locality of interactions that the environment for the region P_k has no effect on the system when the system is in a different region $P_{k'}$, so the conditions of the dispersion entropy theorem are easily and very generally satisfied for the position variable. In any frame that moves significantly with respect to the physical environment, localization is not permanent, because the system will necessarily move from one region to another, spoiling the position localization. So the environment provides the reference frame for position localization.

4. Proofs of the localization theorems

These proofs depend on the fact that Itô equations are 'non-anticipating', so that $MX\,d\xi = 0$ for any X.

First, as a very simple example, which illustrates the main principles of the proofs, let $|\psi\rangle$ be the state vector of a system with zero Hamiltonian, which interacts with its environment through a single environment operator L, so that the state diffusion equation is

$$|d\psi\rangle = (\langle L^\dagger \rangle L - \tfrac{1}{2} L^\dagger L - \tfrac{1}{2} \langle L^\dagger \rangle \langle L \rangle)|\psi\rangle\, dt + (L - \langle L \rangle)|\psi\rangle\, d\xi. \tag{4.1}$$

Suppose that L operates only in the subspace of the projector P, so that

$$L = PL = LP = PLP. \tag{4.2}$$

The system weakly localizes into the space of the projector or its complement, in the sense that the change of the mean square deviation of the projector is less than or equal to zero. To show this, from (1.3), it is enough to show that

$$\mathrm{M}\,\mathrm{d}(\Delta p)^2 = \mathrm{M}\,\mathrm{d}(p - p^2) = \mathrm{M}(1-2p)\,\mathrm{d}p - \mathrm{M}(\mathrm{d}p)^2 \tag{4.3}$$

is not positive.

In the following derivations all terms of order higher than $\mathrm{d}t$ and all fluctuations, with zero mean, of order higher than $(\mathrm{d}t)^{1/2}$, are neglected, so equations (2.4*b*) can be used without the ensemble means. The value of $\mathrm{d}p$ is

$$\begin{aligned}
\mathrm{d}p = \mathrm{d}\langle\psi|P|\psi\rangle \\
&= (\langle\psi|P|\mathrm{d}\psi\rangle + \mathrm{CC}) + \langle\mathrm{d}\psi|P|\mathrm{d}\psi\rangle \\
&= (\langle L^\dagger\rangle\langle L\rangle - \tfrac{1}{2}\langle L^\dagger L\rangle - \tfrac{1}{2}\langle L^\dagger\rangle\langle L\rangle\langle P\rangle + \mathrm{CC}) + ((\langle L\rangle - \langle L\rangle\langle P\rangle)\,\mathrm{d}\xi + \mathrm{CC}) \\
&\quad + \langle((L^\dagger - \langle L^\dagger\rangle)P(L - \langle L\rangle))\,\mathrm{d}t \\
&= \langle\psi|L - \langle L\rangle P|\psi\rangle\,\mathrm{d}\xi + \mathrm{CC} \\
&= (1 - \langle P\rangle)\langle L\rangle\,\mathrm{d}\xi + \mathrm{CC}
\end{aligned} \tag{4.4}$$

where CC means the complex conjugate of the terms to its left.

It follows that, since $\mathrm{M}X\,\mathrm{d}\xi = 0$ for any X,

$$\begin{aligned}
&\mathrm{M}(1-2p)\,\mathrm{d}p = 0 \\
&\mathrm{M}(\mathrm{d}p)^2 = 2|\langle L\rangle|^2(1-p)^2\,\mathrm{d}t \geqslant 0
\end{aligned} \tag{4.5}$$

from which the required result (4.3) follows. Notice that, although the probability over the whole ensemble of being in the P channel cannot be changed by interactions with the environment that couple only states within the channel, for a single member of the ensemble the probability p of being in the subspace almost always changes as a result of such interactions.

Now consider the general localization theorem, in which there is a non-zero Hamiltonian H and there are many orthogonal channels with projection operators P_k satisfying

$$P_k P_\ell = P_\ell P_k = \delta_{k\ell} P_\ell \qquad \sum_k P_k = I. \tag{4.7}$$

The environment operators L_m are block diagonal in the subspaces of the channels, so

$$P_k L_m = L_m P_k = P_k L_m P_k. \tag{4.8}$$

In this case the Hamiltonian term can prevent the localization, so that the mean change in the QMS deviation can have either sign.

The effect of the Hamiltonian is additive and easy to evaluate, and the analysis for the many environment operators follows the same lines as for the simplest case of one environment operator, given by equations (4.1)–(4.5). In particular

$$\mathrm{d}\langle P_k\rangle = (\langle\psi|P_k|\mathrm{d}\psi\rangle + \mathrm{CC}) + \langle\mathrm{d}\psi|P_k|\mathrm{d}\psi\rangle$$

$$
\begin{aligned}
&= \left(\langle\psi| - \frac{\mathrm{i}}{\hbar} P_k H |\psi\rangle + \mathrm{cc}\right) \mathrm{d}t \\
&\quad + \sum_m (\langle\psi| P_k(\langle L_m^\dagger\rangle L_m - \tfrac{1}{2} L_m^\dagger L_m - \tfrac{1}{2}\langle L_m^\dagger\rangle\langle L_m\rangle)|\psi\rangle + \mathrm{cc})\,\mathrm{d}t \\
&\quad + \sum_m (\langle\psi| P_k(L_m - \langle L_m\rangle)|\psi\rangle\,\mathrm{d}\xi_m + \mathrm{cc}) \\
&\quad + \sum_m \langle\psi|(L_m^\dagger - \langle L_m^\dagger\rangle) P_k (L_m - \langle L_m\rangle)|\psi\rangle\,\mathrm{d}t \\
&= -\frac{\mathrm{i}}{\hbar}\langle[P_k, H]\rangle\,\mathrm{d}t \\
&\quad + \sum_m (\langle L_m^\dagger\rangle\langle P_k L_m\rangle - \tfrac{1}{2}\langle P_k L_m^\dagger L_m\rangle - \tfrac{1}{2}\langle P_k\rangle\langle L_m^\dagger\rangle\langle L_m\rangle + \mathrm{cc})\,\mathrm{d}t \\
&\quad + \sum_m ((\langle P_k L_m\rangle - \langle P_k\rangle\langle L_m\rangle)\,\mathrm{d}\xi_m + \mathrm{cc}) \\
&\quad + \sum_m (\langle P_k L_m^\dagger L_m\rangle - \langle L_m^\dagger\rangle\langle P_k L_m\rangle - \langle L_m\rangle\langle P_k L_m^\dagger\rangle + \langle P_k\rangle\langle L_m^\dagger\rangle\langle L_m\rangle)\,\mathrm{d}t \\
&= -\frac{\mathrm{i}}{\hbar}\langle[P_k, H]\rangle\,\mathrm{d}t + \sum_m ((\langle P_k L_m\rangle - \langle P_k\rangle\langle L_m\rangle)\,\mathrm{d}\xi_m + \mathrm{cc})
\end{aligned}
\tag{4.9}
$$

and taking the mean over the ensemble we get the physically evident result

$$
\mathrm{M}\,\mathrm{d}p_k = -\frac{\mathrm{i}}{\hbar}\langle[P_k, H]\rangle\,\mathrm{d}t \tag{4.10}
$$

that the change in the overall mean of the ensemble probability of being in the subspace of P_k is independent of the environment, because the environment operators do not couple the subspaces, and depend on the Hamiltonian terms alone.

Using (4.9), (4.10), (4.3) and (3.11), the final result for the localization in the space of P_k or its complement is

$$
\begin{aligned}
\mathrm{M}\,\mathrm{d}(\Delta p_k)^2 &= \mathrm{M}(1 - 2p_k)\,\mathrm{d}p_k - \mathrm{M}(\mathrm{d}p_k)^2 \\
&= (1 - 2\langle P_k\rangle)\left\langle\frac{\mathrm{i}}{\hbar}[H, P_k]\right\rangle - 2\sum_m |\langle P_k L_m\rangle - \langle P_k\rangle\langle L_m\rangle|^2\,\mathrm{d}t
\end{aligned}
\tag{4.11}
$$

which proves the general localization theorem.

The dispersion entropy theorem follows from the general theorem. To second order in $\mathrm{d}x$,

$$
\mathrm{d}(x\ln x) = (1 + \ln x)\,\mathrm{d}x + (\mathrm{d}x)^2/(2x) \tag{4.12}
$$

but $\mathrm{M}\,\mathrm{d}p_k = 0$, so

$$
\mathrm{M}\,\mathrm{d}(p_k \ln p_k) = \mathrm{M}(\mathrm{d}p_k)^2/(2p_k) \tag{4.13}
$$

and by definition (3.8) of the dispersion entropy

$$
\mathrm{M}\,\mathrm{d}Q = -\sum_k \mathrm{M}(\mathrm{d}p_k)^2/(2p_k). \tag{4.14}
$$

For the special conditions (3.1), (3.2) of the entropy theorem,

$$\mathrm{d}p_k = \sum_j ((1 - p_k)\langle L_{kj}\rangle\, \mathrm{d}\xi_{kj} + \mathrm{cc}) - \sum_{j,k'\neq k} (p_k \langle L_{k'j}\rangle\, \mathrm{d}\xi_{k'j} + \mathrm{cc}) \tag{4.15}$$

Therefore from (2.4*b*)

$$\mathrm{M}(\mathrm{d}p_k)^2 = 2(1 - 2p_k) \sum_j |\langle L_{kj}\rangle|^2\, \mathrm{d}t + 2p_k^2 \sum_{j,k'} |\langle L_{k'j}\rangle|^2\, \mathrm{d}t \tag{4.16}$$

and so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{M}Q &= -\sum_k \left(\frac{1}{p_k} - 2\right) \sum_j |\langle L_{kj}\rangle|^2 - \sum_{j,k'} |\langle L_{k'j}\rangle|^2 \\ &= -\sum_k \frac{1 - p_k}{p_k} R_k \end{aligned} \tag{4.17}$$

which proves the dispersion entropy theorem.

5. Discussion

We have shown how the quantum state diffusion equations of an open system represent the process of localization in a subspace or channel of the state space explicitly. The localization due to the environment increases when the ensemble mean of the QMS deviation or the quantum dispersion entropy of the channel projectors decreases, and the two theorems demonstrate that this always happens unless the change is zero. Explicit formulae for the rates of change are obtained in terms of effective interaction rates. In this model localization is characteristic of the interaction of a system with its environment, whether or not that environment contains measuring apparatus or an observer.

It may seem remarkable that the mean quantum dispersion entropy should behave contrarily to every other physical entropy under the conditions given here. But this is just a consequence of the remarkable properties of quantum mechanics, in which the wave properties of a system, which require it to be extended, are followed by particle properties, in which it is localized as, for example, in the two-slit experiment. The dispersion entropy theorem is a precise mathematical expression of these remarkable properties. The reduction of mean dispersion entropy is consistent with the increase in thermodynamic entropy because ensemble entropies increase as a result of the diffusion of pure quantum states and this more than compensates for the decrease in the mean over the quantum dispersion entropies of the individual pure states of the ensemble.

The Hamiltonian term can increase the mean quantum dispersion entropy, and often does, so in general this entropy can either increase or decrease.

In this picture the localization process appears explicitly, whereas in the usual picture the density operator gives an average over the localized states. The reduction of its off-diagonal elements due to the environment has been associated with the localization process before, as described in considerable detail by Zeh [5], Zurek [6] and their colleagues and collaborators, but they do not allow any modification of the Schrödinger equation and, consequently, cannot represent the physical localization of the state vector of an individual system in one universe explictly.

Depending on the nature of the system and its interaction with the environment, localization may take place with respect to many different variables, but localization in position is particularly important because interactions are localized in position space to a greater extent than for other dynamical variables. This contrasts with the models of Ghirardi, Rimini, Weber, Bell and Pearle [7] in which position localization is a primary property that is introduced by hypothesis from the beginning. It is consistent with the quantum jump models of Dalibard, Castin and Molmer, and of Carmichael [8].

Acknowledgments

We are grateful to H D Zeh and J Dalibard for sending material prior to publication. ICP thanks the UK Science and Engineering Research Council for financial support, and also the Quantum Optics Group of Imperial College London, Theoretical Physics at CERN and the Group of Applied Physics at the University of Geneva for stimulating discussions and hospitality whilst parts of the research were being carried out.

References

[1] Gisin N and Percival I C 1992 The quantum state diffusion model applied to open systems *J. Phys. A: Math. Gen.* **25** 5677–91; 1992 *Phys. Lett.* **167A** 315–18
See also Gisin N and Cibils M 1992 *J. Phys. A: Math. Gen.* **25** 5165–76
[2] Gisin N 1984 *Phys. Rev. Lett.* **52** 1657–60
[3] Gisin N 1989 *Helv. Phys. Acta* **62** 363–71
[4] Diósi L 1988 *J. Phys. A: Math. Gen.* **21** 2885–98; 1988 *Phys. Lett.* **129A** 419–23
[5] Zeh H D 1970 *Found. Phys.* **1** 69–76; 1973 *Found. Phys.* **3** 109–16
Kübler O and Zeh H D 1973 *Ann. Phys.* **76** 405–18
Joos E and Zeh H D 1985 *Z. Phys.* B **59** 223–43
Zeh H D 1992 *Phys. Lett.* A submitted
[6] Zurek W H 1982 *Phys. Rev.* D **26** 1862–80; 1991 *Physics Today* **44** 36–44
[7] Ghirardi G-C, Rimini A and Weber T 1986 *Phys. Rev.* D **34** 470–91
Ghirardi G-C, Pearle P and Rimini A 1990 *Phys. Rev.* A **42** 78–89
Bell J S 1987 *Schrödinger, Centenary of a Polymath* ed C Kilmister (Cambridge: Cambridge University Press)
Bell J S 1987 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press)
[8] Dalibard J, Castin Y and Molmer K 1992 *Phys. Rev. Lett.* **68** 580–3
Molmer K, Castin Y and Dalibard J 1993 Monte-Carlo wavefunction method in quantum optics *J. Opt. Soc. Am.* in press
Carmichael B H J private communication
[9] Gisin N 1984 *Phys. Rev. Lett.* **53** 1776; 1990 *Phys. Lett.* **143A** 1